UNSTEADY FILTRATION OF GROUND WATER IN THE CASE OF A NARROW DRAIN

(NEUSTANOVIUSHAIASIA FILTRATSIIA GRUNTOVYKH VOD V SLUCHAYE UZKOI DRENY)

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This note contains a determination of the position of the ground water level in the case when there exists a narrow drain in the region occupied by the ground water (Fig. 1). The results obtained facilitate the study of drainage problems. It is assumed herein that the region originally occupied by ground water is a semi-infinite plane. Such a statement of the problem is acceptable when the impervious layer lies very far below the surface, while the depth of the narrow drain is comparatively small. The linearized condition at the surface of the ground water which was derived in reference [1] will be assumed here.

We shall study a case of unsteady motion when the narrow drain does not contain any water. We shall assume that the change of the ground water level is small.

Let us establish boundary conditions that will hold at the different segments of the surface occupied by the ground water.

The following boundary condition on the velocity potential holds at segments *BD* and *CD* of the free surface. Here k is the filtration coefficient and m is the porosity.

$$\frac{k}{m}\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial t} = 0 \tag{1}$$

Introduce a new variable τ by means of the formula

$$\tau = \frac{k}{m} t \tag{2}$$

In that case the boundary condition becomes

$$\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial \tau} = 0 \tag{3}$$

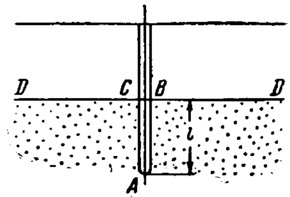

Fig. 1.

At the segments *AB* and *AC* the pressure will be zero during the unsteady motion. The pressure, however, is related to the velocity potential in the following fashion:

$$\varphi = -\left(\frac{p}{\rho g} + ky\right) \tag{4}$$

If $p = 0$ then we obtain $\phi = -ky$. Thus at the segments *AB* and *AC* the following condition holds

$$\varphi = -ky \tag{5}$$

If one introduces a complex velocity potential $w(z) = \phi + i\psi$, then the condition at the free surface may be written as follows

$$\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial \tau} = \operatorname{Re}\left[\frac{\partial w}{\partial \tau} + i\,\frac{\partial w}{\partial z}\right] \tag{6}$$

Introduce now a new function ω, related to the complex velocity potential

$$\omega = \frac{\partial w}{\partial \tau} + i\,\frac{\partial w}{\partial z} \tag{7}$$

In that case, according to (1), the values of its real part are given at the segments *CD* and *BD*. The condition for determining the velocity potential at the segments *AB* and *AC*, which are parallel to the y-axis, is of the form (5). From this it follows that

$$\frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial \tau} = \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial y}\right)(-ky) = -k \tag{8}$$

From (1), (8) and (7) it follows that the real part of the function is known along the entire contour bounded by the lines *Ab*, *AC*, *CD* and *BD*.

Thus, in order to find ω it is necessary to solve the Dirichlet problem for half-plane with a cut subject to boundary conditions as shown in Fig. 2. Above all, it is obvious that the function ω determined in this way will not depend on r.

We shall assume that the depth of the cut, i.e. the depth of the drain, is equal to l. If the half-plane with the cut (the variable z corresponds to this region) is mapped on to a plane which is related to the variable ζ so that points C and B become points with coordinates -1 and $+1$, then the relation between the variables ζ and z will be the following:

$$\zeta = \sqrt{1 + \frac{z^2}{l^2}} \tag{9}$$

The function $\omega(\zeta)$ that satisfies the conditions shown in Fig. 2 will be

$$\omega(\zeta) = -\frac{k}{\pi} \ln \frac{\zeta + 1}{\zeta - 1} \tag{10}$$

In that case, on **the basis** of (9) and (10)

$$\omega(z) = -\frac{k}{\pi} \Lambda(z) \qquad \left(\Lambda(z) = \ln \frac{\sqrt{1 + z^2/l^2} + 1}{\sqrt{1 + z^2/l^2} - 1}\right) \tag{11}$$

Thus we obtain the following differential equation for the determination of $w(z, \tau)$

$$\frac{\partial w}{\partial \tau} + i \frac{\partial w}{\partial z} = \omega = -\frac{k}{\pi} \Lambda(z) \tag{12}$$

We shall assume that originally there was a state of rest in the region occupied by the ground water and that the narrow drain was then created quite rapidly. Therefore, the condition for $w(z, \tau)$ at $\tau = 0$ will be

$$w(z, \tau) = 0 \tag{13}$$

The function $w(z, \tau)$ will be sought in the form of a sum of two components

$$w(z, \tau) = w_0(z) + w_1(z, \tau) \tag{14}$$

Here $w_0(z)$ is the particular solution of equation (12) with the given right-hand side, and $w_1(z, \tau)$ satisfies the homogeneous differential equation

$$\frac{\partial w_1}{\partial \tau} + i \frac{\partial w_1}{\partial z} = 0$$

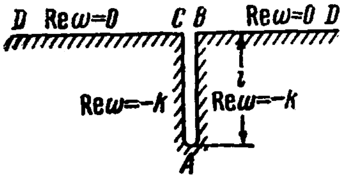

Fig. 2.

The function $w_0(z)$ will not depend on τ. Therefore, it can be found from

$$i \frac{\partial w_0}{\partial z} = -\frac{k}{\pi} \Lambda(z) \quad \text{or} \quad \frac{\partial w_0}{\partial z} = \frac{ki}{\pi} \Lambda(z) \tag{15}$$

Assume that $w_0(z)$ is of the form

$$w_0(z) = \frac{ki}{\pi} \int_0^z \Lambda(\xi)\, d\xi \tag{16}$$

The function $w_1(z)$ that satisfies the homogeneous equation

$$\frac{\partial \omega_1}{\partial \tau} + i \frac{\partial w_1}{\partial z} = 0 \qquad (17)$$

will depend on the combination of variables $z + i\tau$. Consequently, $w_1(z,\tau) = (z + i\tau)$.

In order to have $\tau = 0$, $w(z, \tau) = w_0(z) + w_1(z) = 0$ one has to have

$$w_1(z, 0) = w_0(z) + w_1(z) = 0, \quad w_1(z) = -w_0(z)$$

Then

$$w_1(z, \tau) = -\frac{ki}{\pi} \int\limits_0^{z+i\tau} \Lambda(\xi)\, d\xi \qquad (18)$$

From this we finally get

$$w(z, \tau) = \frac{ki}{\pi}\left[\int\limits_0^{z} \Lambda(\xi)\, d\xi - \int\limits_0^{z+i\tau} \Lambda(\xi)\, d\xi\right] = -\frac{ki}{\pi} \int\limits_z^{z+i\tau} \Lambda(\xi)\, d\xi \qquad (19)$$

By substituting $\Lambda(z)$ from (12) and performing the integration, we obtain

$$w(z, \tau) = \frac{ki}{\pi}\, l \left[\xi \ln \frac{\sqrt{\xi^2+1}+1}{\sqrt{\xi^2+1}-1} - \ln(\xi + \sqrt{\xi^2+1})\right]_{z/l}^{(z+i\tau)/l} \qquad (20)$$

One can determine now from the complex velocity potential the velocity potential ϕ and the velocities of ground water motion at the boundaries of the drain.

The velocity potential is

$$\varphi = \mathrm{Re}\left\{\frac{ki}{\pi}\, l \left[\xi \ln \frac{\sqrt{\xi^2+1}+1}{\sqrt{\xi^2+1}-1} - \ln(\xi + \sqrt{\xi^2+1})\right]_{z/l}^{(z+i\tau)/l}\right\}$$

or

$$\varphi = \frac{kl}{\pi}\, \mathrm{Im} \left[\xi \ln \frac{\sqrt{\xi^2+1}+1}{\sqrt{\xi+1}-1} \ln(\xi + \sqrt{\xi^2+1})\right]_{z/l}^{(z+i\tau)/l} \qquad (21)$$

At the free surface, i.e. at the segments *BD* and *DC* the pressure is $p = 0$. Thus $\phi = -ky$. On the basis of this the change of the ground water level (Fig. 3) is

$$\delta = y = -\frac{1}{k}\varphi$$

Therefore,

$$\delta(x, \tau) = -\frac{l}{\pi}\, \mathrm{Im} \left[\xi \ln \frac{\sqrt{\xi^2+1}+1}{\sqrt{\xi^2+1}-1} - \ln(\xi + \sqrt{\xi^2+1})\right]_{x/l}^{(x+i\tau)/l} \qquad (22)$$

Let us determine the displacement of points at which the free surface is adjacent to the walls of the narrow drain (i.e. the displacement of points C and B). When doing this one has to let $x = 0$. In that case we obtain

$$\delta_{x=0} = -\frac{l}{\pi}\,\mathrm{Im}\left[\frac{i\tau}{l}\ln\frac{\sqrt{1-\tau^2/l^2}+1}{\sqrt{1-\tau^2/l^2}-1} - \ln\left(\frac{i\tau}{2} + \sqrt{1-\frac{\tau^2}{l^2}}\right)\right]$$

or

$$\delta_{x=0} = -\frac{\tau}{\pi}\ln\frac{\sqrt{1-\tau^2/l^2}+1}{\sqrt{1-\tau^2/l^2}-1} + \frac{l}{\pi}\,\mathrm{arc\,tg}\,\frac{\tau/l}{\sqrt{1-\tau^2/l^2}} \qquad (23)$$

Replace the variable τ by the time t by means of formula (2). Then we obtain finally

$$\delta_{x=0} = -\frac{kt}{\pi m}\ln\frac{\sqrt{1-kt/ml}+1}{\sqrt{1-kt/ml}-1} + \frac{l}{\pi}\,\mathrm{arc\,tg}\,\frac{kt/ml}{\sqrt{1-(kt/ml)^2}} \qquad (24)$$

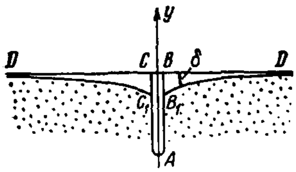

Fig. 3.

This formula will be correct up to the point when

$$\frac{kt}{ml} = 1 \qquad \text{or} \qquad t = \frac{ml}{k} \qquad (25)$$

Then, on the basis of (24), the displacement of a point of the free surface at the boundary of the drain will be

$$\delta_{x=0} = \frac{l}{2}$$

Thus, in the interval of time $t = ml/k$, the boundary of the free surface traverses half the depth of the drain.

When $t < ml/k$ we shall have on the basis of (24)

$$\delta_{x=0} = \frac{2\tau}{\pi}\,\mathrm{arc\,tg}\sqrt{\frac{\tau^2}{l^2}-1} + \frac{l}{2}$$

or passing to the variable t we have

$$\delta_{x=0} = \frac{2}{\pi}\frac{kt}{m}\,\mathrm{arc\,tg}\sqrt{\left(\frac{kt}{ml}\right)^2 - 1} + \frac{l}{2}$$

It should be noted that at values of t of the order of ml/k, the displacements of the ground water level near the wall will be large, which contradicts the original assumption of small changes in water level.

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